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Spaces whose finite sheeted covers are homeomorphic to a fixed space

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Abstract

Finitely generated groups that have the property that all their finite index subgroups are free Abelian are shown to be either free Abelian themselves or to have prime cyclic first homology. This group theoretic result allows one to show that the first homology of a finite connected cell complex that has the property that all of its non-trivial finite index covers have total space homeomorphic to a given space must be either cyclic of prime order or free Abelian. Other topological corollaries include a classification of such 2-complexes and a classification of compact 3-manifolds that have a non-trivial finite sheeted cover and have the property that all their finite-sheeted covers are themselves.

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1. Introduction

In a recent article Miklos [7] asked the following very appealing question.

Question. Suppose you have metric continua X and M having the property that for infinitely many $n \in \mathbb{N}$, n-fold covering projections $f_n : X \to M$ exist. Does it follow that X is homeomorphic to M?

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A moment's reflection shows the answer is no. The Möbius band has $2k$-fold covers by the annulus for all $k \in \mathbb{N}$ and the Klein bottle has $2k$-fold covers by the torus for all $k \in \mathbb{N}$. In neither case is the total space of the covering homeomorphic to the base space. Additional examples can be generated in other ways.

Specifically, higher dimensional examples showing that the answer to the above question is negative can be obtained as follows: Start with a compact connected finite dimensional complex T, (like the circle S^1 or the n-torus $S^1 \times \cdots \times S^1$) which has the property that all its finite sheeted covers have total space homeomorphic to itself. Choose a compact connected finite dimensional complex Y_0 (such as RP^2) that has a non-trivial n_0-fold cover $p : X_0 \to Y_0$ that is not a self-cover. Then for every k-fold self-cover $f_k : T \to T$ the map $F_k : T \times X_0 \to T \times Y_0$ defined by $F_k(t, x) = (f_k(t), p(x))$ is a kn_0-fold cover of $Y = T \times Y_0$ by $X = T \times X_0$. By construction, X and Y are not homeomorphic.

However, even under some restriction the question is of interest. One obvious sort of condition, suggested by the fact that the examples produced in the above paragraphs are of dimension two or greater, comes from restricting to 1-dimensional continua. Some recent work by the second author [9] has focused on the 1-dimensional case. See also [10].

Another line of investigation is suggested by noticing that the examples above have the property that lots of the covering spaces of the given Y have total space homeomorphic to the given X, but there are some that do not. In this case one wonders if the answer might be yes if one were to add the hypothesis that the total space of every finite sheeted covering space in some sufficiently large subcollection of the collection of all finite sheeted covering spaces of Y have total space homeomorphic to X.

In this paper we look at this second question in the context of spaces, such as manifolds or cell complexes, with enough structure that some algebra can be exploited. It seems reasonable to begin by considering only the case where *all* the non-trivial finite sheeted covers of Y have total space homeomorphic to a fixed X. If the answer is still no with this strong hypothesis, it seems likely that little further investigation is warranted.

There is, however, evidence showing that a negative answer will not be obtained in all circumstances. For example, the Classification Theorem for compact connected 2-manifolds implies that the set of all compact connected 2-manifolds Y having the property that all its non-trivial finite sheeted covers have total space homeomorphic to a fixed 2-manifold X can be completely determined. Such a Y is D^2, S^2, RP^2, $S^1 \times I$, or $S^1 \times S^1$. So a closed 2-manifold Y all of whose finite sheeted covers are homeomorphic to X must be either D^2, S^2 or RP^2 or must itself be homeomorphic to X (and both are homeomorphic to either $S^1 \times I$ or $S^1 \times S^1$).

Section 2 contains the group theoretic lemma, Lemma 2.1, that allows us to prove our main topological result. The section also contains several group theoretic corollaries to Lemma 2.1. The proof of the main theorem, Theorem 1.1, and other topological corollaries are in Section 3. The statement of Theorem 1.1 follows.

A covering space $f : X \to Y$ is *non-trivial* if f is an at least 2-to-1 map.

Theorem 1.1. *Let Y be a connected finite complex with a non-trivial finite connected cover. Let X be a connected complex having the property that every non-trivial finite connected covering space of Y has total space homeomorphic to X. Then every finite sheeted cover* *of Y is an Abelian cover with total space homeomorphic to X, $(\pi_1(Y))'$ is a perfect group, and $H_1(Y) \neq 1$. Furthermore either*

(1) *Y has a unique finite sheeted cover that is a p-fold cover for some prime $p \in \mathbb{N}$, and $H_1(Y, \mathbb{Z}) \simeq \mathbb{Z}/p\mathbb{Z}$, or*
(2) *$H_1(Y, \mathbb{Z})$ is finitely generated free Abelian.*

Three of the interesting corollaries to this result are for cases where Y is either a finite regular 2-complex (Corollary 3.3) or a compact orientable 3-manifold (Corollaries 3.6 and 3.8). In each case we show that either $H_1(Y, \mathbb{Z})$ is finite or that X and Y are homeomorphic. In fact, in both of these cases we are able to show that $H_1(Y, \mathbb{Z})$ is finite, or Y is one of the spaces appearing on a short list. Applying this to the case where Y is a compact 3-manifold that is only finitely covered by itself we obtain Corollary 3.8 which states that the only such 3-manifolds are $S^2 \times S^1$, $D^2 \times S^1$, $S^1 \times I \times S^1$, or $S^1 \times S^1 \times S^1$.

Note that Corollary 3.8 is similar to Wang and Wu's classification theorem [12, 8.6] for compact geometric 3-manifolds with a non-trivial self-cover. They prove that a compact 3-manifold that is either geometric or which is non-orientable and is double covered by a geometric 3-manifold non-trivially covers itself if and only if it is covered by a (surface) $\times S^1$ or a torus bundle over S^1. Note that our result does not require the geometric hypothesis employed by Wang and Wu.

2. Groups whose finite index subgroups are finitely generated free Abelian

The main result in this section is the group theoretic lemma, Lemma 2.1, needed to complete the computation of the first homology of the spaces described in Theorem 1.1. We begin with some terminology. First, a finite sheeted covering space will always be connected. Also a finite sheeted covering space is *non-trivial* if it is a k-fold cover for some $k \geqslant 2$. Finally, a subgroup H of a group G is a *proper finite index subgroup* if $\infty > [G : H] \geqslant 2$. Note that this is somewhat non-standard since it allows the trivial subgroup $\langle 1 \rangle \leqslant G$ to be a proper finite index subgroup when G is a finite group. Finally, we define a group G to be *subisomorphic* if all its proper finite index subgroups are isomorphic to each other.

We begin by remarking that it is obvious that a finite group G is subisomorphic if and only if $G \simeq \mathbb{Z}/p\mathbb{Z}$ for some prime $p \in \mathbb{Z}$.

Lemma 2.1. *Let G be a group that has a proper finite index subgroup, then G has the property that all its finite index proper subgroups are finitely generated free Abelian if and only if G is either a finite cyclic group of order p for some prime p, or G is free Abelian.*

Proof. If G is finite, it is of prime order. So we assume that G is infinite. It is immediate that G is finitely generated, so to show that G is free Abelian, it suffices, from the Fundamental Theorem of Finitely Generated Abelian Groups, to show that G is Abelian.

Let H be a proper subgroup of G of finite index. Passing to the normal core of H if necessary, we may assume without loss that H is normal in G.

Suppose that H is contained in two distinct maximal subgroups M and N of G. By assumption, M and N are both finitely generated free Abelian, and by maximality, $G = \langle M, N\rangle$. Let $Z(G)$ denote the center of G, then $H \subseteq M \cap N \subseteq Z(G)$ and $Z(G)$ is of finite index in G; in particular, $Z(G)$ is free Abelian. A result of Schur, see, for example [5, 10.1.4], shows that G', the commutator group of G, is finite. Note that $G' \cdot Z(G)$ is of finite index in G. But if Z is any proper finite index subgroup of $Z(G)$, then $G' \cdot Z$ is a proper finite index subgroup of G, hence free Abelian, which forces $G' = 1$, and gives the required result.

We may now assume that every proper finite index normal subgroup of G lies in a unique maximal subgroup.

In particular, G/H is cyclic of prime power order. If the rank of H is greater than 1, then G has quotients which are not cyclic of prime power order. So $H \simeq \mathbb{Z}$ and every subgroup of H is normal in G and again G has quotients which are not of prime power order. □

Corollary 2.2. *If G is soluble, subisomorphic, and has a finitely generated proper finite index subgroup, then G is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ or G is finitely generated free Abelian.*

Proof. Let H be a finitely generated proper finite index subgroup of G. Since H is a subgroup of a soluble group, it is also soluble. Either H has a proper finite index subgroup or it does not. If H has no proper finite index subgroup, then the fact that H is soluble implies that H is the trivial group. So, $G \simeq \mathbb{Z}/p\mathbb{Z}$. Suppose then that H has a proper finite index subgroup. Then, by [6, 2.1], it follows that H is finitely generated free Abelian. Since every finite index subgroup of G is isomorphic to H it follows that every finite index subgroup of G is finitely generated free Abelian. So, by Lemma 2.1, it follows that G is finitely generated free Abelian. □

Corollary 2.3. *If G is residually finite, subisomorphic, and has a finitely generated proper finite index subgroup, then G is free Abelian of rank at least* 1 *or $G \simeq \mathbb{Z}/p\mathbb{Z}$ for some prime p.*

Proof. Let H be a finitely generated proper finite index subgroup of G. A finite index subgroup of a residually finite group is residually finite. So H is residually finite. If H has no proper finite index subgroup it follows that $H = \{1\}$ and so $G \simeq \mathbb{Z}/p\mathbb{Z}$ for some prime p. Now assume that H has a proper finite index subgroup. Then, by [6, 3.4], H is free Abelian. But every finite index subgroup of G is isomorphic to H. So, by Lemma 2.1, G is finitely generated free Abelian. □

3. The proof of Theorem 1.1

The *finite residual* of a group G, denoted $\mathcal{F}(G)$, is defined to be the intersection of all the finite index subgroups of G. Note that $\mathcal{F}(G)$ is normal in G.

Lemma 3.1. *If S is a finite index subgroup of G then $\mathcal{F}(S) = \mathcal{F}(G)$.*

Proof. Suppose that S is a proper finite index subgroup of G. Since $|G:S| < \infty$, then $\{N\colon N < S \text{ and } |S:N| < \infty\} \subset \{N\colon N < G \text{ and } |G:N| < \infty\}$. So $\mathcal{F}(S) \supset \mathcal{F}(G)$. Now suppose that $g \in \mathcal{F}(S)$. Let N be a finite index subgroup of G. Since N is of finite index in G, it follows that $S \cap N$ is of finite index in S. So by definition of $\mathcal{F}(S)$, $g \in S \cap N$. In particular, $g \in N$. Since N is an arbitrary finite index subgroup of G, it follows that $g \in N$ for every finite index subgroup of G. So $g \in \mathcal{F}(G)$. Therefore $\mathcal{F}(S) \subset \mathcal{F}(G)$ and the proof of the claim is complete. $\square$

Lemma 3.2. *Let G be a finitely generated subisomorphic group with a proper finite index subgroup H. If H has no proper finite index subgroups, then H is the unique finite index subgroup of G, $G' = H$ and $G/H \simeq \mathbb{Z}/p\mathbb{Z}$. If H has a proper finite index subgroup then $\mathcal{F}(G) = \mathcal{F}(H) = H' = G'$, $G/G' \neq 1$, and G/G' is finitely generated free Abelian.*

Proof. We begin by considering the case where H has no proper finite index subgroups. In this case it is clear that H is then the unique finite index subgroup of G, for if N were another finite index subgroup, then $H \cap N$ would be of finite index in both H and N, from which it follows that $N = N \cap H = H$. Since H is the unique finite index subgroup in G, it follows that $H \lhd G$. Thus G/H is a finite group with no non-trivial proper finite index subgroups and $G/H \simeq \mathbb{Z}/p\mathbb{Z}$ for some prime p. So $H \supset G'$. Since H is finitely generated and has no proper finite index subgroups, it follows that $H = G'$.

Now suppose that H has a proper finite index subgroup. We may assume without loss of generality that $H \lhd G$; if it is not, pass to the normal core of H which is also of finite index in G. The normal core will still have a proper finite index subgroup since it is isomorphic to H. Since every proper finite index subgroup of G is isomorphic to H, and since every finite index subgroup of H is also of finite index in G, it follows that every finite index subgroup of H is isomorphic to H, that is, in the terminology of [6], H is a finitely generated hc-group. So, by [6, 3.1], H/H' is finitely generated free Abelian, H' is a perfect group, and $H' = \mathcal{F}(H)$. Observe that since H has a proper finite index subgroup, the rank of the free Abelian group H/H' is at least 1.

Since H is normal in G, it follows that $\mathcal{F}(H) = H' \lhd G$. We now consider H' and G/H'. We want to show for every finite index subgroup $S \subset G$ that $H' = S'$.

From Lemma 3.1 it follows for every finite index subgroup S of G that $\mathcal{F}(S) = \mathcal{F}(G) = \mathcal{F}(H)$. Since G is subisomorphic, $S' = \mathcal{F}(S) = \mathcal{F}(H) = H' = \mathcal{F}(G)$. In particular, every finite index subgroup $S/\mathcal{F}(G)$ of $G/\mathcal{F}(G)$ is free Abelian. Therefore, by Lemma 2.1, $G/\mathcal{F}(G)$ is finitely generated free Abelian. Observe that $\mathcal{F}(G) \subset G'$. So $G/\mathcal{F}(G)$ Abelian implies that $\mathcal{F}(G) = G'$. The proof of Lemma 3.2 is complete. $\square$

Observe that Theorem 1.1 is an immediate corollary of Lemma 3.2, depending only on the correspondence between subgroups of the fundamental group of nice spaces, such as connected compact manifolds or cell complexes, and covering spaces of the base space. The remainder of this section explores additional topological corollaries. We begin with some terminology and some elementary facts about spaces that have the property that all their finite sheeted covers have total space homeomorphic to themselves and some elementary facts about spaces that have all their proper finite sheeted covers homeomorphic to a fixed space.

A connected topological space X has the *finite self-cover property* (FSP) if every connected finite sheeted covering space of X has total space homeomorphic to X. A space with FSP will be called *trivial* if it has no non-trivial finite sheeted covers, and *non-trivial* otherwise. A space Y has the *constant finite cover property* (CFCP) if every non-trivial finite sheeted cover of Y has total space homeomorphic to a fixed space X, called the finite sheeted cover of Y. Spaces with FSP are also called *h-connected spaces* in the literature.

Now we list some elementary results on spaces with self-covers, spaces with FSP, and spaces with CFCP. First, a finite connected n-dimensional cell complex X that has a non-trivial finite sheeted self-cover has Euler characteristic equal to 0, $\pi_1(X)$ is infinite, and the singular set of X, that is, the set of points of X that do not have neighborhoods homeomorphic to the n-ball, must also have non-trivial self covers. In particular, if X is a manifold with boundary and X has a non-trivial self-cover, then ∂X also has a non-trivial self-cover. It follows that the only 2-manifolds that can appear as components of the boundary of a compact 3-manifold with a non-trivial self-cover are $S^1 \times S^1$ and the Klein bottle. Note that if X is a cell complex that has FSP and is non-trivial, then $H_1(X, \mathbb{Z}) \neq 0$. (We show this in the proof of Corollary 3.8.) A manifold that has a non-trivial finite sheeted cover and has the finite self-cover property must be orientable since it has an orientable double cover. Also note that if Y is a cell complex with CFCP then its finite sheeted covers have FSP. When the finite sheeted cover of Y has FSP and is non-trivial, then $\chi(Y) = 0$.

Finally observe that if a connected finite 1-complex K has the property that all its non-trivial finite sheeted covers are homeomorphic to a fixed space, X, then K must be homeomorphic to X and each must be either a finite tree or S^1. The remainder of the results in this section deal with finite 2-complexes or compact 3-manifolds that possess the covering space properties of interest.

Corollary 3.3. *Let K be a finite regular 2-complex with CFCP and with a non-trivial finite-sheeted cover X. Then either*

(1) *X has no non-trivial finite sheeted covers, so that K has a unique non-trivial finite sheeted cover with total space homeomorphic to X; $\pi_1(X)$ has no proper finite index subgroups; there is a prime p such that $\pi_1(K)$ is an extension of $\pi_1(X)$ by $\mathbb{Z}/p\mathbb{Z}$; and $H_1(K, \mathbb{Z}) \simeq \mathbb{Z}/p\mathbb{Z}$, or*
(2) *X has FSP and is non-trivial and either $\pi_1(K) \simeq \mathbb{Z}$ and there is a finite tree T such that K is homeomorphic to $S^1 \times T$, or $\pi_1(K) \simeq \mathbb{Z} \oplus \mathbb{Z}$ and K is homeomorphic to $S^1 \times S^1$. In either case K is homeomorphic to X.*

Proof. As K has the property that all its non-trivial finite sheeted covers are homeomorphic to X, it follows that X has FSP. Either X is trivial or non-trivial.

We begin by considering the case where X is trivial. In this case, Theorem 1.1 implies that K has a unique finite sheeted cover X, $\pi_1(X)$ has no non-trivial finite index subgroups and $\pi_1(K)$ has a unique finite index subgroup with index larger than 1. Note that the specified finite index subgroup is isomorphic to $\pi_1(X)$, it is the commutator subgroup of $\pi_1(K)$, and there is a prime p such that $H_1(K, \mathbb{Z}) \simeq \mathbb{Z}/p\mathbb{Z}$ for some prime p.

In the case where X is non-trivial [8, 3.10] implies that X is homeomorphic to either $S^1 \times T$ for some finite tree T, or X is homeomorphic to $S^1 \times S^1$. In either case, it follows that every finite index subgroup of $\pi_1(K)$ is finitely generated free Abelian (of, respectively, rank 1 or 2) and so by Lemma 2.1 it follows that $\pi_1(K)$ is finitely generated free Abelian of, respectively, rank 1 or 2.

When X is the 2-torus, the only possible choices for K are the 2-torus or the Klein bottle. But the Klein bottle has $(2k+1)$-fold self-covers for all positive integers k and the Klein bottle is not homeomorphic to the 2-torus. So when X is the 2-torus it follows that K is also the 2-torus. When X is $S^1 \times T$ for some finite tree, it follows that K is a locally trivial fiber bundle over S^1 with fiber T. If K were itself not a product, then it would also have finite-sheeted self-covers, namely those of order congruent to 1 modulo the order of the characteristic map, which would necessarily not be homeomorphic to X. So when X is $S^1 \times T$, it follows that K must also be an $S^1 \times T$. □

Corollary 3.4. *If M is a compact, connected,* 3*-manifold with CFCP whose prime factors are either virtually Haken, or have finite or cyclic fundamental group, then $\pi_1(M)$ is isomorphic to one of* 1*, $\mathbb{Z}/p\mathbb{Z}$ for some prime p, $\mathbb{Z}$, $\mathbb{Z} \oplus \mathbb{Z}$, or $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.*

Proof. By Hempel [2], every 3-manifold M satisfying the hypotheses on connect summands given in the statement of this corollary has a residually finite fundamental group. Let X be such that the total space of every non-trivial finite-sheeted cover of M is homeomorphic to X. Then $\pi_1(M)$ has the property that every proper finite index subgroup is isomorphic to $\pi_1(X)$. Therefore, by Corollary 2.3, $\pi_1(M)$ is finitely generated free Abelian or is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for some prime p. By Hempel [1, 9.13] or Jaco [3, V.8], the only finitely generated free Abelian groups that can be fundamental groups of 3-manifolds are 1, $\mathbb{Z}$, $\mathbb{Z} \oplus \mathbb{Z}$, or $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. □

Observe that for each of the groups listed in Corollary 3.4 there is a compact 3-manifold with the given group as fundamental group that satisfies the hypotheses of the corollary. For a representative space whose fundamental group is the trivial group, choose S^3; for a representative space whose fundamental group is $\mathbb{Z}/p\mathbb{Z}$, choose a lens space $L(p,1)$; and for the groups $\mathbb{Z}$, $\mathbb{Z} \oplus \mathbb{Z}$, and $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ choose, respectively, $S^1 \times D^2$, $S^1 \times S^1 \times I$, and $S^1 \times S^1 \times S^1$. An example of a closed 3-manifold that has the finite self-cover property and whose fundamental group is $\mathbb{Z}$ is $S^2 \times S^1$.

Corollary 3.5. *If the connected reducible* 3*-manifold M has CFCP, then either M is homeomorphic to $S^2 \times S^1$ or $H_1(M) = 1$ or $\mathbb{Z}/p\mathbb{Z}$ for some prime p.*

Proof. Suppose that $H_1(M)$ is non-trivial, is not a finite cyclic p-group for any prime p, and M is not irreducible. Since $H_1(M)$ is not finite, Theorem 1.1 implies that $H_1(M)$ is free Abelian of rank at least 1. Also since M is not irreducible, there is an $S_0^2 \subset M$ such that S_0^2 does not bound a 3-cell in M.

Case 1. For this case we suppose that S_0^2 separates and derive a contradiction, thereby showing that this case cannot occur. We have that $M = U \# V$, for some 3-manifolds U and V. Since $H_1(M)$ has a $\mathbb{Z}$-summand, M has regular cyclic k-fold covers for each $k \in \mathbb{N}$. In fact, the Meyer–Vietoris sequence for (M, U, V) implies that we may assume without loss of generality that $H_1(U)$ is infinite. Therefore, U has k-fold covers U_k, for each $k \geqslant 1$.

These extend to k-fold covers $M_k = U_k\#V\#\cdots\#V$ (k copies of V) of M. So each M_k is a 3-manifold that is a non-trivial connect sum that non-trivially covers itself. By [11], each M_k is a $P^3\#P^3$. But $P^3\#P^3$ has non-trivial finite covers by $S^2 \times S^1$. This contradicts the fact that N has CFCP. So Case 1 cannot occur.

Case 2. Now assume that S_0^2 is a non-separating 2-sphere in M. By Jaco [3, II.1], $M = M_1\#(S^2 \times_\varphi S^1)$ where $S^2 \times_\varphi S^1$ is an S^2-bundle over S^1 with sewing map φ. If $M_1 \neq S^3$, then the sphere defining the connect sum is a separating reducing sphere, which has been ruled out in Case 1.

Hence, $M_1 = S^3$ and $M = S^2 \times_\varphi S^1$. If the gluing map φ of the bundle is orientation reversing, it follows that M has some non-trivial odd order self-covers and M has even order self covers by $S^2 \times S^1$. So the case where φ is orientation reversing is impossible. So it follows that when S_0^2 is non-separating, $M = S^2 \times S^1$. The proof of Corollary 3.5 is complete. □

Corollary 3.6. *If M is an orientable Haken manifold with CFCP, then $M = D^3$, $S^1 \times D^2$, $S^1 \times S^1 \times I$, or $S^1 \times S^1 \times S^1$.*

Proof. It is well-known that an orientable Haken manifold is D^3 or has infinite fundamental group. The proof of this is simple: If $\partial M \neq \emptyset$, then the irreducibility of M implies that either $M = D^3$ or ∂M has positive genus and hence M has positive first Betti number. If M is closed, it contains a 2-sided closed incompressible surface, which must be orientable and has positive genus, hence $\pi_1(M)$ is infinite.

By Theorem 1.1 and the above, it suffices to show that if N is a Haken manifold with $\pi_1(N) = \mathbb{Z}$, $\mathbb{Z}^2$, or $\mathbb{Z}^3$, then it is homeomorphic to one of the manifolds M listed in the corollary.

It is well-known that a Haken manifold is aspherical, i.e., it is a $K\pi_1$. Therefore if N is a Haken manifold with the same fundamental group as M, then there is a map $f : N \to M$ which induces an isomorphism on the fundamental groups. Let $M = S^1 \times D^2$, $S^1 \times S^1 \times I$, or $S^1 \times S^1 \times S^1$. Then up to homotopy we may assume that $f(\partial N) \subseteq \partial M$. (This is trivial when $M = S^1 \times S^1 \times S^1$, and is easy for the other manifolds because then M can be deformed into ∂M by a homotopy.) By [3, Theorem X.8], if there is a map $f : (N, \partial N) \to (M, \partial M)$ between Haken manifolds which induces an isomorphism on the fundamental group then N is homeomorphic to M. Hence the result follows. □

Corollary 3.7. *If M is a non-orientable compact* 3*-manifold with CFCP, then $H_1(M) = \mathbb{Z}/2\mathbb{Z}$.*

Proof. We have shown that if $H_1(M)$ is infinite, then M is double covered by $S^1 \times F$ for $F = S^2$, D^2, T^2, or $S^1 \times I$. But each of these spaces is Seifert fibered and hence geometric. So by [12, 8.4] M admits a self-cover. Hence it does not have CFCP (since it also has an orientable double cover) unless it has a unique finite sheeted cover. But since M has a double cover, this and Theorem 1.1 imply that $H_1(M) = \mathbb{Z}/2\mathbb{Z}$. □

Corollary 3.8. *If M is a compact* 3*-manifold that has the FSP and is non-trivial, then M is homeomorphic to $S^2 \times S^1$, $D^2 \times S^1$, $S^1 \times I \times S^1$, or $S^1 \times S^1 \times S^1$.*

Proof. Let $G = \pi_1(M)$. Since M has a non-trivial finite cover, it follows that G has a proper finite index subgroup, H. By passing to the normal core of H if necessary, we may assume without loss of generality that H is normal in G and G/H is a non-trivial finite group. Let $Hx \in G/H$ be an element of minimal prime order p. Let $\langle Hx\rangle^*$ be the inverse image of the cyclic subgroup $\langle Hx\rangle$ under $G \to G/H$. Note that since M has FSP, it follows that every finite index subgroup of G is isomorphic to G. So $\langle Hx\rangle^* \simeq G$. But $|\langle Hx\rangle^* : H| = p$ so G itself has a normal subgroup of index p. But by Theorem 1.1, $H_1(M)$ is free Abelian, so $H_1(M)$ has a $\mathbb{Z}$-summand.

This implies that M has a 2-fold self-cover and is therefore orientable. Also, since M has a $\mathbb{Z}$-summand in its first homology, M contains a two-sided incompressible surface by Jaco [3, III.10]. If M is not irreducible, then Corollary 3.5 implies that M is homeomorphic to $S^2 \times S^1$. If M is irreducible, then Corollary 3.6 implies that M is one of the other three spaces listed in the statement of this corollary. □

4. Questions

Question 1. It is conjectured, see Hempel [2] or Kirby [4, 3.5], that all 3-manifolds have a connect sum decomposition into summands that are either virtually Haken, or have finite or cyclic fundamental group. Hence, assuming that the conjecture is true, Corollary 3.4 is presumably a result about all 3-manifolds and, as a consequence, Corollaries 3.4 and 3.6 would imply that there are only four orientable 3-manifolds with the constant finite sheeted cover property with infinite fundamental groups. The consequence of the conjecture that is directly responsible for the conclusions in Corollaries 3.4 and 3.6 is the fact that all such 3-manifolds must have residually finite fundamental group. We ask if all 3-manifolds with the CFCP can be shown to have residually finite fundamental group in a manner that is independent of this conjecture.

Question 2. In [3, X.5] Jaco asks if a homotopy RP^3 must be homeomorphic to RP^3. The answer to this question has direct bearing on any attempt to deal with the classification of 3-manifolds with CFCP that have finite first homology. Here is a more general version of Jaco's question. We know of three compact 3-manifolds with π_1 isomorphic to $\mathbb{Z}/2\mathbb{Z}$. They are RP^3, $RP^2 \times I$, and the twisted I-bundle over RP^2, i.e., the space $S^2 \times I/\sim$ where $(x,t) \sim (y,s)$ if and only if $t = s \neq 0$ and $x = y$, or if $t = s = 0$ and $y = \pm x$. Are there any other compact 3-manifolds with π_1 isomorphic to $\mathbb{Z}/2\mathbb{Z}$?

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